[Discrete Mathematics]

Notes for graphs and recurrence relations

Whilst I put these notes up on my site to help others, it takes me about quarter of an hour to convert 20 scans (even with an ADF scanner) into PDFs. I cannot guarantee that I will continue doing this.

Jamie Balfour

Recurrence relations (RR)

Linear RR with constant coefficients:

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + C_k a_{n-k} + f(n)$$

when $f(n) = 0$ it is called homogeneous RR.
If $f(n) \neq 0$ it is called a non homogeneous.

Examples

$$a_n = 7a_{n-1} - 12a_{n-2} \qquad \text{hom}$$

$$a_n = a_{n-1} + 2 \times a_{n-2} + 4 \times 3^n \qquad \text{non}$$

$$a_n = a_{n-1} + 5_n \qquad \text{non}$$

Arithmetic sequence

$$a_n = a_{n-1} + C \qquad \text{a non-homo}$$

$a_0 = 1 = 1$ $\qquad C = 2$
$a_1 = 1 + 2 = 3$
$a_2 = 3 + 2 = 5$
$a_3 = 5 + 2 = 7$

$$\boxed{a_n = a_0 + C \cdot n} \qquad \text{a general solution}$$

General solution contains parameters (a_0, C) to be fixed if initial conditions are given.

For $a_0 = 1$, $C = 2$ $a_n = 1 + 2n$ is the solution

Geometric Sequence

$a_n = r \times a_{n-1}$ homogeneous RR

eg $r = 2$, $a_0 = 1$

$a_1 = 2 \times 1 = 2$, $a_2 = 2 \cdot 2 = 4$

$\boxed{a_n = a_0 \cdot r^n}$ general solution

eg for $r = 2$, $a_0 = 1$ $a_n = 2^n$ is the solution

Find a general solution to the RR $a_n = 5a_{n-1}$

$a_n = a_0 \times 5^n$ general solution

OR

$a_n = C \cdot 5^n$ (just another way of writing this)

Find the solution to $a_n = 5a_{n-1}$

With $a_1 = 20$.

Start with general solution $a_n = C \cdot 5^n$

Sub $n = 1$

$$a_1 = C \cdot 5^1 = 20 \quad \text{solve this to get } C = 4$$

Substitute into the general solution to solve the final solution:

$$a_n = 4 \cdot 5^n$$

Problem find a general solution to the RR

$$a_n = 7a_{n-1} - 12a_{n-2}$$

Need 2 terms, eg a_0, a_1 to generate numbers
$\Rightarrow$ the generatisolution will contain 2 unknown constants

Let's look for a solution of

$$a_n = C \cdot r^n$$

$$a_{n-1} = C \cdot r^{n-1}$$

$$a_{n-2} = C \cdot r^{n-2}$$

Substitute into the RR

$$C \cdot r^n = 7 \cdot C \cdot r^{n-1} - 12 \cdot C \cdot r^{n-2}$$

Can assume $C \neq 0$, $r \neq 0$ so divide by C and by the smallest power r^{n-2}

$$\frac{r^n}{r^{n-2}} \to r^2 = 7 \cdot \frac{r^{n-1}}{r^{n-2}} = r - 12 \times \cancel{\frac{r^{n-2}}{r^{n-2}}}$$

$r^2 = 7r - 12$ so we've got a quadratic equation

$$\boxed{\begin{array}{l} ax^2 + bx + c = 0 \\ x = \dfrac{b \pm \sqrt{b^2 - 4ac}}{2a} \end{array}}$$

No n-dependence!

$r^2 - 7r + 12 = 0$

$\therefore \quad r = 7 \pm 12 = 0$

$$r = \frac{7 \pm \sqrt{(-7)^2 - 4 \times 12}}{2} = \frac{7 \pm 1}{2}$$

$r_1 = 4$, $r_2 = 3$ two solution

$a_n = C \cdot 4^n$ is a solution

$a_n = C \cdot 3^n$ is another solution

$$\boxed{a_n = C_1 \cdot 4^n + C_2 \cdot 3^n}$$ general solution

General method that works for any homogeneous RR

$$a_n = p \cdot a_{n-1} + q \cdot a_{n-2}$$

$$r^2 = p \cdot r + q$$

If r_1, r_2 are the roots of quadratic the general solution has the formula

$$a_n = C_1 \cdot r_1^n + C_2 \cdot r_2^n$$

Problem Find the solution to $a_n = 7a_{n-1} - 12a_{n-2}$ with $a_1 = 1$ and $a_2 = 7$

1st find general solution
2nd sub initial conditions into the general solution

$n=1 \quad 1 = C_1 \cdot 4^1 + C_2 \cdot 3^1$

$n=2 \quad 7 = C_1 \cdot 4^2 + C_2 \cdot 3^2$

We next solve this system of simultaneous eqs. for C_1, C_2

$4C_1 + 3C_2 = 1 \Rightarrow C_2 = \left(\frac{1-4C_1}{3}\right)$

Sub into eq_2

$7 = C_1 \cdot 16 + 9 \cdot \left(\frac{1-4C_1}{3}\right)$

$7 = C_1(16-12) + 3$

$4 = C_1 \cdot 4 \Rightarrow C_1 = 1$

$\Rightarrow C_2 = \frac{1-4\cdot 1}{3} = -1$

Substitute the values of C_1 and C_2 into general solution:

$$a_n = 4^n - 3^n$$

Fibonacci number satisfy

$$a_n = a_{n-1} + a_{n-2}$$

$r^2 = r + 1$

$r^2 - r - 1 = 0$

$r = \frac{1 + \sqrt{1-4(-1)}}{2} = \frac{1+\sqrt{5}}{2}$

$$a_n = C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$a_0 = 1, a_1 = 1$ We get

$$C_1 = \frac{1}{2}\left(1+\frac{1}{\sqrt{5}}\right) \qquad C_2 = \frac{1}{2}\left(1-\frac{1}{\sqrt{5}}\right)$$

<u>Example 1</u>

Robot can move in steps of 1, 2 or 3 meters.

Q. In how many ways can it cover a distance of n meters.

$n = \underline{1} \cdot \underline{1}$, $n\ 2 \cdot 2$, $n = 3 \cdot 4$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

# of different ways

last step 1n

last step 2n

last step 3n

Introduction to modular arithmetic

$| 25, \ 25$

$a, b, \in \mathbb{Z} \setminus \{0\}$

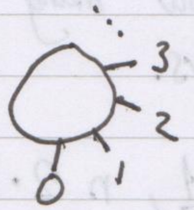
a divides b $(a \mid b)$

if there is a $c \in \mathbb{Z}$ such that $a \cdot c = b$

If not then there is a $r \in \mathbb{Z}$ such that
$b = q \cdot c + r$

With $0 < r < a$

r: remainder



$b = r \bmod a$. $(b \equiv r \bmod a)$

Example $102 = 2 \bmod 25$

Arithmetics of remainders: modular arithmetics

<u>Properties:</u> 1) $(a+b) \bmod m$

$$= [(a \bmod m) + (b \bmod m)] \bmod m$$

eg. $102 \bmod 25 = 2$
$36 \bmod 25 = 11$

$138 \bmod 25 = [(2 \bmod 25) + (11 \bmod 25)] \bmod 25$

$= 13 \bmod 25$
$= 13$

2. $(a \cdot b) \bmod m = [(a \bmod m) \cdot (b \bmod m)] \bmod m$

eg $(102 \cdot 36) \bmod 25 = [(102 \bmod 25)(36 \bmod 25)] \bmod 25$

$= [2 \cdot 11] \bmod 25$

$= 22 \bmod 25$
$= 22$

3. $a^n \bmod m = [(a \bmod m)^n] \bmod m$

{from part 2}

e.g. $13 \bmod 6 = 1$

$13^{2014} \bmod 6 = [(13 \bmod 6)^{2014}] \bmod 6$
$= [1^{2014}] \bmod 6$
$= 1 \bmod 6$
$= 1$

<u>Prime number decomposition</u>

An integer $p > 1$ is called a prime number if it is only divisible by 1 and p.
Fundamental theorem of arithmetics;

Gaussian elimination

m - equations on n unknowns

$a_{11} x_1 + a_{12} x_2$

$a_{m1} x_1 + \ldots$

The augmented matrix

$$(A \mid B) = \begin{array}{cccc} x_1 & x_2 & \ldots & x_n \\ \left(\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{matrix} \right. & \begin{matrix} a_{12} \\ a_{22} \\ \\ a_{m2} \end{matrix} & \begin{matrix} \ldots \\ \ldots \\ \\ \ldots \end{matrix} & \left. \begin{matrix} a_{1n} \\ a_{2n} \\ \\ a_{mn} \end{matrix} \right) \end{array}$$

This | is used because A and B work differently

Multiply equations by numbers and adding them together is equivelant to performing <u>elementary row operations</u> on the rows of the augmented matrix.

The row operations are denoted as

1) $R_j \longrightarrow R_j + K \cdot R_i$

Means add K (a number) times row i to j

2) $R_i \longrightarrow k \cdot R_i$ means multiply row i by the number k.

To eliminate the coefficients in the first column we use the top element to annihilate the elements below it.

A coefficient used to annihilate the entries below it is called a <u>pivot</u>. The process of eliminating all entries below a pivot may be referred to as <u>downsweeping</u>.

Gaussian elimination (GE):

1) downsweep in the 1^{st} column below the 1^{st} row, then in the 2^{nd} row and so on.

2) Find the solution by back substitutions starting from the bottom equation and going upward.

<u>Example</u> Solve using GE the system

$$\begin{aligned} x + y + z &= 1 \\ 3x + 4y - z &= 2 \\ -2x + y - z &= 8 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1^* & 1 & 1 & 1 \\ 3 & 4 & -1 & 2 \\ -2 & 1 & -1 & 8 \end{array}\right) \qquad R_2 \to R_3 - 3R_1$$

$$\left(\begin{array}{ccc|c} 1^* & 1 & 1 & 1 \\ 0 & 1 & -4 & -1 \\ -2 & 1 & -1 & 8 \end{array}\right) \qquad R_3 \rightarrow R_3 + 2R_1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1^* & -4 & -1 \\ 0 & 3 & 1 & 10 \end{array}\right) \qquad R_3 \rightarrow R_3 - 3R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 13 & 13 \end{array}\right)$$

As we have gone diagonally as far as possible, [13 would be the last pivot, yet it has nothing diagonally right from it] we have finished

sub $z=1$

2) $y - y \cdot z = -1$

$y - 4 = -1 \rightarrow y = 3$

1) $x + y + z = 1$

$x + 3 + \underline{1} = 1 \rightarrow x = 3$

The unique solution is

$$(x, y, z) = (-3, 3, 1)$$

Example: Solve

$$x+2y+3z=1$$
$$4x+5y+6z=2$$
$$7x+8y+9z=1$$

$$\left(\begin{array}{ccc|c} 1^* & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 1 \end{array}\right)$$

$R_2 \to R_2 - 4R_1$
$R_3 \to R_3 - 7R_1$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3^* & -6 & -2 \\ 0 & -6 & -12 & -6 \end{array}\right)$$

$R_3 \to R_3 - 2R_2$

We only downsweep $-6, -12$ as our new pivot is R_2, C_2

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -2 \\ 0 & 0 & 0 & -2 \end{array}\right)$$

The bottom equation is

$$0 \cdot x + 0 \cdot y + 0 \cdot z = -2$$

No solution [s]!

Example

For this example, one should take the pevious matrix for referrence such that:

$$\left(\begin{array}{ccc|c} 1^* & 2 & 3 & 1 \\ 2 & 4 & 5 & 1 \\ -1 & 1 & 2 & 4 \end{array}\right) \qquad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0^* & -1 & -1 \\ 0 & 3 & 5 & 5 \end{array}\right)$$

We swap rows

$$\boxed{R_2 \Leftrightarrow R_3}$$

The boxed text infers we have switched two rows:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 0 & -1 & -1 \end{array}\right)$$

If the natural pivotal element is 0 find a non-vanishing element below it, swap the corresponding rows. But what happens if the pivot is still 0?

Example

$$\left(\begin{array}{ccc|c} 1^* & 2 & 3 & 1 \\ 2 & 4 & 5 & 1 \\ -1 & -2 & 2 & 4 \end{array}\right) \qquad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0^* & -1 & -1 \\ 0 & 0 & 5 & 5 \end{array}\right)$$

Oh look: The pivot is 0 and swapping rows would still be 0. If this happens, move the pivot directly to the right. In this case -1 $[R_2, C_3]$ becomes the pivot.

$$R_3 \rightarrow R_3 + 5 \cdot R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Here we can see 2 different equations for 3 variables. Therefore there is no unique solution.

In such solutions/situations we split the variables into free dependent ones.

y, x, z

$-z = -1 \Rightarrow z = 1$

$x + 2y + 3z = 1$

$x + 2y + 3 = 1$

$x = -2 - 2y$

<u>General solution</u> contains some arbitrary constant

$$(x, y, z) = (-2-2y, y, 1)$$