Discrete Mathematics

Notes by

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[DISCRETE MATHEMATICS]

Whilst I put these notes up on my site to help others, it takes me about quarter of an hour to convert 20 scans (even with an ADF scanner) into PDFs. I cannot guarantee that I will continue doing this.

Set theory

Definition: A set is an <u>unordered</u> collection of <u>distinct</u> objects

$A = \{a, b, d, g\}$

A is a set which contains 4 letters.

$B = \{x \mid x \text{ is a positive eken number less than } 7\}$

such that (can also be represented as :)

equilelantly $B = \{2, 4, 6\}$

1) $\{2, 4, 2\}$ is not a set

2) Order of elements does not matter

$x \in b$ = x is an element of b

$B \subset A$ = B is as subset of A. You can use $\subseteq$

$A = \{1, \{3,5\}, 3, 6, 7, a\}$ 6 elements

$3 \in A$, $\{3,5\} \in A$, $5 \notin A$,

$\{3,6\} \subseteq A$, $\{3\} \subset A$

$|A|$ = cardinality of set A (the number of elements in A)

$\mathbb{Z}$ = set of all integers

$\mathbb{N}$ = set of natural numbers 1, 2, 3...

The union of sets: $A \cup B = \{x \mid x \in A \text{ \underline{or} } x \in B\}$

The intersection of sets: $A \cap B = \{x \mid x \in A \text{ \underline{and} } x \in B\}$

Venn diagrams

Union Intersects

$A - B$ - in A and not in B

Empty set: $\emptyset$

If the universal set U is specified one can define a complement of a set for any set $A \subset U$

$\overline{A} = \{x \mid x \notin A\}$

Computer Representation of sets

We can represent any subset of U by a bit string.

1 in k^{th} position means $U_k \in A$
0 in k^{th} position means $U_k \notin A$

Mathematical induction (M.I)

Example 1

Consider the sums of consequtive odd numbers

$1+3=4=2^2$
$1+3+5=9=3^2$
$1+3+5+7=16=4^2$

For any integer $n \geq 1$

$1+3+5+7 \ldots + (2n-1) = n^2$

Check For $n=4$ $\quad 2n=8 \quad 8-1=7$

M.I. $P(n): 1+3+5+\ldots+(2n-1)=n^2$

Suppose that the formula holds for some particular but unspecified $n=k$

Assume $P(k)$ is true. We would like to show that this implies that $P(k+1)$ is true.

$P(k): 1+3+5+\ldots+(2k-1)=k^2$

$P(k+1): \underbrace{1+3+5+\ldots+(2k-1)}_{k^2} + (2(k+1)-1)$

$=(k+1)^2$

By $P(k)$

$1+3+\ldots+(2k-1)+(2k+1) = k^2+2k+1$
$=(k+1)$

$\boxed{\begin{array}{c}P(k)\\ \text{True}\end{array}} \rightarrow \boxed{\begin{array}{c}P(k+1)\\ \text{True}\end{array}}$

- In proofs by M.I. we start by doing the <u>base step</u> $P(1)$ is true (n $P(4)$ as in the example)

Induction step:

Assume $P(k)$ is true. We must use this to show that $P(k+1)$ is also true.

<u>Another example</u>

Prove that for any integer $n \geq 1$ $3^n - 1$ is divisible by 2.

<u>Base step</u> $3^1 - 1 = 2$, therefore is divisible by 2 ($2 \% 2 = 0$) so true

<u>Induction step</u>

Assume $3^k - 1$ $2 \times a$ Can use for some integer

Want to show $3^{k+1} - 1 \stackrel{?}{=} 2 * b$

$$3^k = 3^1 \times 3 - 1 = (1 + 2a) \times 3 - 1$$
$$= 3 + 6a - 1 = 2 + 6a = 2 \times (1 + 3a)$$

Is divisible by 2 so $P(k+1)$ is true. By MI $P(n)$ is true for all $n \geq 1$. $\therefore$ This is true.

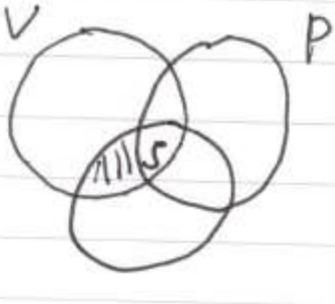
$$\{ n : \mathbb{Z} \mid n = 7 \} = \{7\}$$
$$\{ n : \mathbb{Z} \mid n \in 1..3 \} = \{1, 2, 3\}$$
$$\{ h : \mathbb{N} \mid h < 6 \} = \{1, 2, 3, 4, 5, 6\}$$

a) $|V \cap J \cap P| = ?$

Using IE formula $|V \cup J \cup P| = |V| + |J| + |P| - |V \cap J| - |J \cap P| - |V \cap P| + |V \cap J \cap P|$

$= 70 = 50 + 60 + 30 - 45 - 20 - 10$

b)



5 students from part A.

We need the shaded section

$\underbrace{|V \cap J|}_{45} - \underbrace{|V \cap J \cap P|}_{5} = 45 - 5 = 40$

$$|S \cup F \cup R| = |F| + |S| + |R| - |F \cap S| - |F \cap R| - |R \cap S| + |R \cap S \cap F|$$

$$= 200 + 150 + 60 - 90 - 20 - 30 + 3$$
$$= 273$$

Example: Among 1st year CS students all have to one of 3 languages: VB, Java, Perl

The are 70 1st years chose
VB = 50, Java = 60, Perl = 30

45 study both Java & VB
20 study both Perl & VB
10 study both Perl & Java

a) Find the number of students who study all 3 languages.

b) Find the number of students will be the number of students who study VB & Java but not Perl.

Denote V = set of students who study VB, J for Java and P for Perl.

$$|V \cup J \cup P| = 70$$

$|V| = 50$, $|J| = 60$, $|P| = 30$, $|V \cap J| = 45$,

$|P \cap V| = 20$, $|P \cap J| = 10$

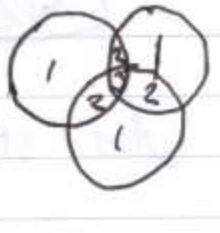
$\left[\frac{500}{3}\right] = [166.66\ldots] = 166$

$|A| = 166 \quad |B| = \left[\frac{500}{13}\right] = 38$

We are after $|A \cup B| = |A| + |B| - |A \cap B|$

$A \cap B$ are numbers between 1 and 500 divisible by $3 \cdot 13 = 39$

$|A \cap B| = \left[\frac{500}{39}\right] = [12.82\ldots] = 12$

A  B

$|A \cup B \cup C| =$
$|A| + |B| + |C| -$
$|A \cap B| - |A \cap C| - |B \cap C| +$
$|A \cap B \cap C|$

<u>Example 1</u> 200 students have taken Spanish, 150 - French, 60 - Russian, 30 took both F & R, 90 took both F & S, 20 took both S & R and 3 took all.

$|S| = 200 \quad |F| = 150 \quad |R| = 60$
$|S \cap F| = 90 \quad |S \cap R| = 20 \quad |R \cap F| = 30$
$|S \cap F \cap R| = 3$

Corrects the counting sum rule.

Example 1

Find the number of cards which are clubs ~~and~~ or aces.
Let C be the set of all clubs $|C| = 13$. Let A be the set of all aces. $|A| = 4$.

$A \cap C$ contains only one card - the ace of clubs.

$$|A \cup C| = |A| + |C| - |A \cap C|$$

$$= 4 + 13 - 1 = 16$$

Only one of the four aces is within the clubs.

Example 2

How many integers from 1 to 500 are divisible by 3 or by 13?

Let $A = \{ n \mid 1 \leq n \leq 500,\ n = 3 \cdot m \}$
$B = \{ n \mid 1 \leq n \leq 500,\ n = 13 \cdot k \}$

$$A = \left\{ 3 \cdot 1, 3 \cdot 2, 3 \cdot 3, \ldots 3 \cdot \left[\frac{500}{3}\right] \right\}$$

Here [ ] denotes the integer part

Example

A full house in cards/poker is a collection of 5 cards in which 3 of them are from one denomonation and 2 from another. There are 13 denominations with 4 cards each. How many full houses are there?

$$13 \text{ cards} \times \binom{4!}{3!} \times 12 \text{ cards} \times \binom{4!}{2!} =$$

$$13 \times \binom{4}{3} \times 12 \times \binom{4}{2} = 13 \cdot 4 \cdot 12 \cdot 6$$

$$= 3744$$

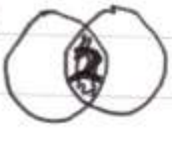
choices for denom. for 3 cards

denom for 2. cards

The inclusion-exclusion formula

The I-E formula for two sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

A 

$$|A| + |B| - |A \cap B|$$

Some properties: $$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Pascal's triangle

$n=0$ 1

$n=1$ 1 1

$n=2$ 1 2 1

$n=3$ 1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

<u>Example</u> 52 cards. A hand is 13 cards. How many possible hands are there? Order doesn't matter and there is no repetition.

So we have combinations:

$$\binom{52}{13} = \frac{52!}{13!\,39!} = 635{,}013{,}559{,}600$$

$+ \binom{5}{2} x^2 y^3 + \binom{5}{3} x^3 y^2 + \binom{5}{4} x^4 y^1$

$+ \binom{5}{5} x^5 y^0 = y^5 + 5xy^4 + 10x^2y^2$

$+ 10x^3y^2 + 5x^4y + x^5$

$\binom{5}{0} = \frac{5!}{5!0!} = 1, \binom{5}{1} = \frac{5!}{4!1!} = 5$

$\binom{5}{2} = \frac{5!}{3!2!} = 10$

$\underbrace{(x+y)\cdot(x+y)\dots(x+y)}_{n \text{ times}}$

$\boxed{\text{What the fuck?!}}$

For $x^k y^{n-k}$ we have to pick x k times and y $n-k$ times. Picking k brackets out of n = the number of combinations $\binom{n}{k}$

Definition A combination of n objects taken k at any time is any selection of k distinct ~~elemen~~ objects (no repetitions). Combination is thus an unordered selection without repetitions.

Theorem The number of distinct combinations of n objects of n objects taken k at a time is:

$$\binom{n}{k} = \frac{n!}{n-k!\,k!}$$

Proof : We know the answer for permutations. To pick a permutation we first pick a combination and then we pick an order by the product rule for counting

$$\# \text{ of perm } \frac{n!}{(n-k)!} = (\# \text{ of combinations}) \cdot k!$$

$$\rightarrow \# \text{ of combs} = \frac{n!}{(n-k)!\,k!}$$

$\binom{n}{k}$ are called binomial coefficients because they appear in the binomial formula:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k\, y^{n-k}$$

eg. $n = 5$

$$(x+y)^5 = \binom{5}{0} x^0 y^5 + \binom{5}{1} x^1 \cdot y^4$$

Repititions are not allowed. 1, B, B is not a permutation

The order matters so that 1, 3, B and B, 1, 3 are different permutations.

We will often say "ordered selection without repititions"

Example In how many ways can a committee of 10 people select between themselves a treasurer, a chairperson and an admin.

Order 1st is chairman, 2nd is treasurer, 3rd is admin.

No repetitions → we have a permutation of 3 people out of 10.

$10 \times 9 \times 8 = 720$

Theorem: The number of ordered selections of k distinct objects from a set of n objects (permutations) is

$$n(n-1)\cdot(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

$$\frac{1\cdot 2\ldots(n-k)\cdot(n-k+1)\cdot(n-k+2)\ldots n}{1\cdot 2\cdot 3\ldots(n-k)}$$

NB When $k=n$ we have the number of ways to order n distinct elements: $n!$

Example

3 letter words in lower case letters if

a) Any letter can be used in any position
$26 \times 26 \times 26 = 17,576$

b) The letters must be distinct
$26 \times 25 \times 24 = 15,600$

c) The first and last letters must be consonants.
$21 \times 5 \times 21$

d) The first ~~or~~ last letter must be consonants.

$21 * 5 * 1$
or (+) $= 210$
$5 * 1 * 21$

Permutations and combinations

Definition of permutation

A permutation of n objects is any ordered selection of R distinct (no repetitions) objects from a given set of n objects.

Example $\{A, B, 1, 3\}$ a set of 4 objects

The lists 1, 3, B
A, 1, 3
B, 1, 3
permutations of the 4 given objects taken 3 at a time